THE STRUCTURE OF THE PERIODIC SOLUTIONS OF A QUASI-LINEAR SELF-CONTAINED SYSTEM WITH SEVERAL DEGREES OF FREEDOM IN THE CASE OF DIFFERING, BUT PARTLY NONCOMMENSURATE FREQUENCIES

PMM Vol. 32, No. 3, 1968, pp. 548-552

A.P. PROSKURIAKOV
(Moscow)

(*Received February 15, 1968*)

The present paper concerns the structure of the periodic solutions of a quasilinear self-contained system in the case where some of the frequencies of the generating system are commensurate with each other but not commensurate with the other frequencies. We shall show that the functions constituting the quasinormal coordinates of the system break down into two groups, each of which has its own characteristic properties.

Let us consider the following quasilinear self-contained system with n degrees of freedom:

$$\sum_{k=1}^{n} (a_{ik}x_k^{\cdot\cdot} + c_{ik}x_k) = \mu F_i(x_1, \ldots, x_n, x_1^{\cdot}, \ldots, x_n^{\cdot}, \mu) \qquad (i = 1, \ldots, n)$$

$$a_{ik} = a_{ki}, \ c_{ik} = c_{ki} \tag{1}$$

We assume that the functions $F_i\ (x_k, x_k^{\cdot}, \mu)$ are analytic in x_k, $x_k^{\cdot}$ and in the small parameter μ in the ranges of x_k and $x_k^{\cdot}$ for $0 < \mu < \mu_0$.

Let the equation of the generating system frequencies ($\mu = 0$),

$$\Delta\ (\omega^2) = |\ c_{ik} - \omega^2 a_{ik}\ | = 0 \tag{2}$$

have all its roots distinct and positive. In [1] we showed that the general structure of an arbitrary (not necessarily periodic) solution of the quasilinear system is in this case of the form

$$x_k(t) = \sum_{r=1}^{n} p_k^{(r)} x^{(r)}(t) \qquad (k = 1, \ldots, n) \tag{3}$$

The coefficients of the form $p_k^{(r)}$ can be expressed in terms of the algebraic complements of the corresponding elements of determinant (2),

$$p_1^{(r)} = 1, \qquad p_k^{(r)} = \frac{\Delta_{ik}(\omega_r^2)}{\Delta_{i1}(\omega_r^2)} \qquad (k = 2, \ldots, n) \tag{4}$$

The functions $x^{(r)}(t)$ are a generalization of the normal coordinates of the linear generating system which they become for $\mu = 0$. We have the following general expression for $x^{(r)}(t)$:

$$x^{(r)}(t) = (A_r + \beta_r) \cos \omega_r t + \frac{B_r + \gamma_r}{\omega_r} \sin \omega_r t + \tag{5}$$

$$+\sum_{m=1}^{\infty} C_m^{(r)}(t, A_1+\beta_1, \ldots, A_n+\beta_n, B_1+\gamma_1, \ldots, B_n+\gamma_n)\mu^m \qquad (r=1,\ldots,n)$$

The functions $C_m^{(r)}$ appearing in this expression are given by Formula

$$C_m^{(r)}(t) = \Big[\Delta_0\omega_r \prod_{s=1}^{n}{}' (\omega_s^2-\omega_r^2)\Big]^{-1} \int_0^t R_m^{(r)}(t')\sin\omega_r(t-t')\,dt' \qquad (6)$$

The prime next to the product symbol means that the value $s = r$ of the subscript is omitted. Moreover, we have

$$\Delta_0 = |a_{ik}| \qquad (7)$$

$$R_m^{(r)}(t) = \sum_{i=1}^{n} \Delta_{i1}(\omega_r^2) H_{im}(t), \quad H_{im}(t) = \frac{1}{(m-1)!}\left(\frac{d^{m-1}F_i}{d\mu^{m-1}}\right)_{\beta_s=\gamma_s=\mu=0}$$

Let us assume that not all of the frequencies ω_r are commensurate. For example, let the first l frequencies be commensurate with each other, but not commensurate with the remaining $n-l$ frequencies ($l<n$). Let us consider the structure of the periodic solution of such a system in more detail.

From the general solution of the generating system, which in this case happens to be nonperiodic, we isolate the periodic solution with the period T_0 corresponding to the frequencies $\omega_1, \ldots, \omega_l$. This imposes the following conditions on the initial data of the generating system: $A_r = 0$, $B_r = 0$ for $r = l+1, \ldots, n$.

By virtue of the fact that the system is self-contained we can assume that $B_1 = 0$, $\gamma_1 = 0$.

Thus, the initial conditions for the functions $x^{(r)}(t)$ entering into the periodic solution of system (1) under consideration and for their first derivatives with respect to the time t are

$$\begin{aligned} &x^{(r)}(0) = A_r + \beta_r \qquad (r=1,\ldots,l)\\ &x^{\cdot(1)}(0) = 0,\ x^{\cdot(r)}(0) = B_r+\gamma_r \qquad (r=2,\ldots,l)\\ &x^{(r)}(0) = \beta_r,\ \dot{x}^{(r)}(0) = \gamma_r \qquad (r=l+1,\ldots,n)\end{aligned} \qquad (8)$$

For $\mu \neq 0$ this periodic solution has the period $T = T_0 + \alpha$, where α is a function of μ which vanishes for $\mu = 0$.

We transform the time in initial system (1) in such a way that the period of the solution of system (1) becomes equal to T_0 and therefore independent of the parameter μ. The transformation is of the form

$$t = \tau h = \tau(1 + h_1\mu + \ldots)$$

The form of the subsequent terms of the series for h is determined by the character of the expansions of β_r and γ_r in whole or fractional powers of the parameter μ. This character is in turn contingent on the multiplicity of the solutions of the amplitude equations. The above time transformation does not alter the preceding formulas and also preserves the values of the initial data under which the system has periodic solutions with the period T_0. The functions $x^{(r)}(t)$ are replaced by the functions $x^{(r)*}(\tau)$,

$$x^{(r)}(t) = x^{(r)*}(\tau) = (A_r+\beta_r)\cos\omega_r\tau + \frac{B_r+\gamma_r}{\omega_r}\sin\omega_r\tau +$$

$$+\sum_{m=1}^{\infty} C_m^{(r)*}(\tau, A_1+\beta_1, \ldots, A_n+\beta_n, B_1+\gamma_1, \ldots, B_n+\gamma_n)\mu^m \qquad (9)$$

$$B_1 = 0, \quad \gamma_1 = 0, \quad A_r = B_r = 0 \qquad (r = l+1, \ldots n)$$

If all of the parameters β_r and γ_r can be expanded in series in whole powers of μ, i.e. if

$$\beta_r = \sum_{m=1}^{\infty} A_{rm}\mu^m, \qquad \gamma_r = \sum_{m=1}^{\infty} B_{rm}\mu^m$$

then the quantity h can be represented by means of a similar series. The functions $C^{(r)*}(\tau)$ are then defined by Formulas

$$C_1^{(r)*}(\tau) = C_1^{(r)}(\tau) + A_{r1}\cos\omega_r\tau + \frac{B_{r1}}{\omega_r}\sin\omega_r\tau - h_1\tau(\omega_r A_r\sin\omega_r\tau - B_r\cos\omega_r\tau) \quad (10)$$

$$\begin{aligned} C_2^{(r)*}(\tau) = {} & C_2^{(r)}(\tau) + A_{r2}\cos\omega_r\tau + \frac{B_{r2}}{\omega_r}\sin\omega_r\tau + h_1\tau C_1'^{(r)}(\tau) - \\ & - \tau\left[\omega_r(h_1A_{r1} + h_2A_r)\sin\omega_r\tau - (h_1B_{r1} + h_2B_r)\cos\omega_r\tau\right] - \\ & - \frac{1}{2}h_1^2\omega_r\tau^2(\omega_rA_r\cos\omega_r\tau + B_r\sin\omega_r\tau) \text{ и т. д.} \end{aligned} \quad (11)$$

Construction of the periodic solutions of system (1) is possible if and only if all of the functions $x^{(r)*}(\tau)$ are periodic. The conditions of periodicity of these functions are the relations

$$\begin{aligned} & x^{(r)*}(T_0) = A_r + \beta_r \qquad (r = 1, \ldots, l) \\ & x^{(1)*\prime}(T_0) = 0, \; x^{(r)*\prime}(T_0) = B_r + \gamma_r \qquad (r = 2, \ldots, l) \\ & x^{(r)*}(T_0) = \beta_r, \; x^{(r)*\prime}(T_0) = \gamma_r \qquad (r = l+1, \ldots, n) \end{aligned} \quad (12)$$

By virtue of the above considerations, the functions $x^{(r)*}(\tau)$ can be expanded in whole powers of μ,

$$x^{(r)*}(\tau) = x_0^{(r)*}(\tau) + \mu x_1^{(r)*}(\tau) + \mu^2 x_2^{(r)*}(\tau) + \ldots \quad (13)$$

with coefficients having the period T_0,

$$\begin{aligned} & x_0^{(r)*}(\tau) = A_r\cos\omega_r\tau + \frac{B_r}{\omega_r}\sin\omega_r\tau \\ & x_1^{(r)*}(\tau) = A_{r1}\cos\omega_r\tau + \frac{B_{r1}}{\omega_r}\sin\omega_r\tau + C_1^{(r)*}(\tau) \\ & x_2^{(r)*}(\tau) = A_{r2}\cos\omega_r\tau + \frac{B_{r2}}{\omega_r}\sin\omega_r\tau + C_2^{(r)*}(\tau) + \\ & + \sum_{s=1}^{n}\left(\frac{\partial C_1^{(r)*}(\tau)}{\partial A_s}A_{r1} + \frac{\partial C_1^{(r)*}(\tau)}{\partial B_s}B_{r1}\right) \quad \text{etc.} \end{aligned} \quad (14)$$

We note that the first two coefficients $x_q^{(r)*}(\tau)$ do not contain derivatives of the functions $C_m^{(r)*}(\tau)$ with respect to A_s and B_s. These derivatives first appear in the coefficient $x_2^{(r)*}(\tau)$.

Let us show that the functions $C_m^{(r)*}(\tau)$ are independent of β_s and γ_s for all r and m provided that $s = l+1, \ldots, n$. To do this we need merely show that the derivatives of these functions with respect to β_s and γ_s are identically equal to zero for $s = l+1, \ldots, n$. Instead of the derivatives with respect to β_s and γ_s it will be more convenient for us to compute the derivatives with respect to A_s and B_s, since all of the indicated functions depend only on the sums $A_s + \beta_s$ and $B_s + \gamma_s$. For this reason the derivatives with respect to A_s and B_s do not lose meaning even when A_s and B_s are known to equal zero. But if it is also the case that $\beta_s = 0$ and $\gamma_s = 0$, the derivatives with respect to A_s and B_s computed in this way have no sense and must be equated to zero. Conversely, if the formally computed derivatives with respect to A_s and B_s for $A_s = 0$, $B_s = 0$ contradict conditions imposed on the functions in which they appear as terms (e.g. the periodicity conditions), then the assumption concerning the dependence of the functions $C_m^{(r)*}(\tau)$ on β_s and γ_s is invalid.

Let us analyze the structure of the terms entering into the coefficients $x_2^{(r)*}(\tau)$ of expansions (13). By virtue of the amplitude equations, all of the functions $C_1^{(r)*}(\tau)$ become periodic for $r = 1, \ldots, l$; for $r = l+1, \ldots, n$ they have the structure [2]

$$C_1^{(r)*}(\tau) = \Phi_1^{(r)}(\tau) + P_1^{(r)}\cos\omega_r\tau + Q_1^{(r)}\sin\omega_r\tau \quad (15)$$

Here and below the $\Phi_m^{(r)}(\tau)$ denote periodic functions of τ with the period T_0, while the quantities $P_m^{(r)}$ and $Q_m^{(r)}$ remain constant. We recall that the frequencies ω_r are noncommensurate with the frequency ω_0 for $r = l+1, \ldots, n$.

The functions $C_2^{(r)}(\tau)$ for $r = 1,\dots, l$ are of the form

$$C_2^{(r)*}(\tau) = \Phi_2^{(r)}(\tau) + P_2^{(r)}\tau \cos \omega_r\tau + Q_2^{(r)}\tau \sin \omega_r\tau \tag{16}$$

For $r = l + 1,\dots, n$ their structure is defined by Formula (15).

Now let us consider the derivatives of $C_1^{(r)*}(\tau)$ with respect to A_s and B_s. These derivatives are of the form (16) for $r, s = 1,\dots, l$, and of the form (15) for $r = l + 1,\dots, n$ and $s = 1$. As we noted above, all of the formulas defining the functions $C_m^{(r)}(t)$ remain valid for the functions $C_m^{(r)*}(\tau)$. Let us compute the derivative of $R_1^{(r)*}(\tau)$ with respect to A_s for $s = l + 1,\dots, n$ making use of Formulas (3), (4), and (7). We have

$$\frac{\partial R_1^{(r)*}(\tau)}{\partial A_s} = \sum_{i=1}^{n}\sum_{k=1}^{n}\left[\Delta_{ik}(\omega_r^2)\left(\frac{\partial H_{i1}}{\partial x_{..0}}\cos\omega_s\tau - \frac{\partial H_{i1}}{\partial x_{.0}'}\sin\omega_s\tau\right)\right]$$

After integrating we have

$$\frac{\partial C_1^{(r)*}(\tau)}{\partial A_s} = \Phi_3^{(r)}(\tau)\cos\omega_s\tau + \Phi_4^{(r)}(\tau)\sin\omega_s\tau + P_3^{(r)}\cos\omega_r\tau + Q_3^{(r)}\sin\omega_r\tau$$
$$(r = 1, \dots, n;\ s = l + 1, \dots, n) \tag{17}$$

The derivatives of $C_1^{(r)}(\tau)$ with respect to B_s for $s = l + 1,\dots, n$ are of the same form. The structure of these derivatives differs considerably from that of the remaining terms entering into the coefficients $x_2^{(r)*}(\tau)$. Hence, the periodicity of these coefficients can be ensured only if the indicated derivatives are equal to zero. This means that the assumption that the functions $C_1^{(r)*}(\tau)$ depend on β_s and γ_s which we made in computing these derivatives was not valid. The nondependence of these functions on β_s and γ_s for $s = l + 1,\dots, n$ has therefore been proved. Hence, summation of the last terms in the formula for the coefficients $x_2^{(r)*}(\tau)$ must be carried out from $s = 1$ to $s = l$.

In the same way, by considering the coefficients $x_3^{(r)*}(\tau)$ we can prove the nondependence of the functions $C_2^{(r)*}(\tau)$ on β_s and γ_s for $s = l + 1,\dots, n$, etc.

Let us now show that the parameters β_s and γ_s for $s = l + 1,\dots, n$ are analytic functions of the initial functions $x^{(r)*}(\tau)$ for $r = 1,\dots, l$, and of the parameter μ. This property was first arrived at by Malkin [3] for a quasilinear non-selfcontained system consisting of first-order equations [3].

From the conditions of periodicity for any function $x^{(s)*}(\tau)$ and for its first derivative with respect to τ we obtain

$$\beta_s(\cos\omega_s T_0 - 1) + \frac{\gamma_s}{\omega_s}\sin\omega_s T_0 + \sum_{m=1}^{\infty} C_m^{(s)}(T_0, A_1 + \beta_1, \dots, B_l + \gamma_l)\mu^m = 0$$
$$(s = l + 1, \dots, n) \tag{18}$$
$$-\beta_s\omega_s\sin\omega_s T_0 + \gamma_s(\cos\omega_s T_0 - 1) + \sum_{m=1}^{\infty} C_m'^{(s)}(T_0, A_1 + \beta_1, \dots, B_l + \gamma_l)\mu^m = 0$$

Each pair of such relations can be regarded as a system of two equations in β_s and γ_s. Since the determinant of this pair of equations is

$$\Delta_s = (\cos\omega_s T_0 - 1)^2 + \sin^2\omega_s T_0 \neq 0$$

the systems are solvable. The parameters β_s and γ_s are analytic functions of $A_r + \beta_r$ and $B_r + \gamma_r$ for $r = 1,\dots, l$ and of the parameter μ,

$$\begin{aligned}\beta_s &= \varphi_{s-l}(A_1 + \beta_1, \dots, A_l + \beta_l, B_2 + \gamma_2, \dots, B_l + \gamma_l, \mu)\\ \gamma_s &= \psi_{s-l}(A_1 + \beta_1, \dots, A_l + \beta_l, B_2 + \gamma_2, \dots, B_l + \gamma_l, \mu)\end{aligned} \qquad (s = l + 1, \dots n) \tag{19}$$

The functions ϕ_{s-l} and ψ_{s-l} vanish for $\mu = 0$.

The results of our analysis imply that the functions $x^{(r)*}(\tau)$ appearing in the periodic solution of system (1) can be broken down into two groups in the case of distinct but partially noncommensurate frequencies of the generating system. The first group consists of functions with the subscripts $r = 1,\dots, l$, i.e. with the subscripts of the commensurate frequencies associated with the periodic solution with the period T_0. These functions are analytic functions of the initial data of their group and of the parameter μ. The second group consists of functions with the subscripts $r = l + 1, \ldots, n$ which are also analytic functions of the initial data of the first-group functions and of the parameter μ; the latter are, in addition, linear functions of the characteristic initial data β_r and γ_r. These are, in turn, analytic functions of the initial data of the first group and of the parameter μ; they vanish for $\mu = 0$.

Returning to the functions $C_m^{(r)}(t)$, we readily infer from Formula (9) that these functions are also independent of β_s and γ_s if $s = l + 1, \ldots, n$. The above properties of the functions $x^{(r)*}(t)$ and of their initial data enable us to devise a simple method of constructing the periodic solutions of quasilinear self-contained systems with distinct but partially noncommensurate frequencies of the generating solution. In the case of two degrees of freedom this method is described in [2].

We note that the structure of the periodic solutions of a quasilinear non-selfcontained system consisting of n second-order equations is considered in [4 and 5].

The authors of the above papers did not show that the functions $C_m^{(r)}(t)$ are independent of β_s and γ_s for $s = l + 1, \ldots, n$, even though they assume in several of their formulas the derivatives of these functions with respect to β_s and γ_s equal zero.

BIBLIOGRAPHY

1. Proskuriakov, A.P., On a property of periodic solutions of quasilinear autonomous systems with several degrees of freedom. PMM Vol. 24, No. 4, 1960.
2. Proskuriakov, A.P., On the construction of periodic solutions of quasi-linear autonomous systems with several degrees of freedom. PMM Vol. 26, No. 2, 1962.
3. Malkin, I.G., The methods of Liapunov and Poincaré in the Theory of Nonlinear Oscillations. Gostekhizdat, Moscow-Leningrad, 1949.
4. Plotnikova, G.V. and Bainov, D.D., The periodic oscillations of a mechanical system with n degrees of freedom in the presence of resonance frequencies. Rev. Roumaine Sci. Tech. Sér. Méch. Appl. Vol. 10, No. 2, 1965.
5. Bainov, D.D., Periodic oscillations of a non-selfcontained quasilinear system with n degrees of freedom in the presence of resonance frequencies. Izv. mat. Inst. Bulg. Akad. Nauk Vol. 9, 1966.

Translated by A.Y.